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On the topology of manifolds with completely integrable geodesic flows II

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We show that if M is a compact simply connected Riemannian manifold whose geodesic flow is completely integrable with non-degenerate first integrals, then the loop space homology of M grows sub-exponentially. We also show that if for some point $p \in M$, the geodesic flow of M admits action–angle coordinates with singularities in a neighborhood of every vector in the unit sphere at p, then M is $\mathbb{Z}$-elliptic.

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1. Introduction

In this paper we continue the investigation we began in [21–23] concerning topological properties of manifolds with completely integrable geodesic flows.

It is a classical result of Dinaburg [4] that the vanishing of the topological entropy h_{top} of the geodesic flow ψ_t of a compact Riemannian manifold M implies that the fundamental group, $\pi_1(M)$, grows sub-exponentially; but perhaps more interesting, if $\pi_1(M)$ is finite then it follows from results of Gromov and Yomdin (cf. [21, Theorem 3.2]), that $h_{\text{top}} = 0$ implies that the loop space homology of the manifold grows sub-exponentially. In particular M is rationally elliptic (cf. [12]); a very restrictive property. Thus a natural approach to the problem of finding topological restrictions on manifolds with completely integrable geodesic flows is to investigate the vanishing of the topological entropy of ψ_t. In a more general setting the relevant question is: does the topological entropy of a completely integrable Hamiltonian vanish? In [21,22] we showed

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that the topological entropy vanishes in the case of a collective integrable system [13] (collective complete integrability includes the Thimm method [26]), and it is the purpose of the present paper to investigate further this matter.

We will show that if the first integrals verify a rather "generic" condition, then the topological entropy of the Hamiltonian flow on any compact invariant set vanishes (cf. Theorem 2.2). We will also show that if for some point $p \in M$, the geodesic flow of M admits action–angle coordinates with singularities in a neighborhood of every vector in the unit sphere at p, then M is $\mathbb{Z}$-elliptic [9] (cf. Theorem 2.5).

2. Definitions and statement of results

We will always assume that all Hamiltonian vector fields are complete and that all our objects are C^∞.

Let (X, ω) be a symplectic $2n$-dimensional manifold with Poisson bracket $\{\ ,\ \}$ and let $H\colon X \to \mathbb{R}$ a smooth (i.e. C^∞) Hamiltonian with associated Hamiltonian vector field ξ_H. The Hamiltonian H is said to be *completely integrable* if there exist n smooth functions (called *first integrals*) $f_1, \dots, f_n$ defined on X so that:

(i) $\{f_i, f_j\} = 0 \quad i, j = 1, \dots, n$,

(ii) $\{f_i, H\} = 0 \quad i = 1, \dots, n$,

(iii) the functions $f_1, \dots, f_n$ are functionally independent almost everywhere, i.e., there exists an open set $X' \subset X$ whose complement has zero measure so that $df_1, \dots, df_n$ are linearly independent for each point in X'.

Frequently one finds in the literature a definition of completely integrability that differs sligthly from the previous one in the sense that in condition (iii) one only asks for the set X' to be dense. In the real analytic category the two conditions are equivalent.

Set now $F := (f_1, \dots, f_n) : X \to \mathbb{R}^n$. As it is well known the flow of ξ_H leaves the level sets $F^{-1}(c)$ invariant and if c is a regular value of F and $F^{-1}(c)$ is compact, then the connected components of $F^{-1}(c)$ are n-dimensional tori T^n and in a neighborhood of these tori there exists a canonical coordinate system (I, θ), $\theta \in T^n$, $I \in \mathbb{R}^n$ such that H is a function only of the n variables $I = (I_1, \dots, I_n)$. This implies that the flow of ξ_H is periodic or quasi-periodic on the n-tori defined by $I =$ const. The coordinates (I, θ) are called *action–angle* coordinates. The previous result was proved by Arnold [2] under certain conditions, later removed by Jost [16] (see also [1,6,20]).

A crucial assumption for the existence of action–angle coordinates is that c is a regular value of F. One could ask then : how is the dynamics of ξ_H on the singular set i.e, on the complement of X'? Can the flow of ξ_H generate positive topological entropy on the singular set? For arbitrary completely integrable smooth Hamiltonians there is, a priori, no reason to expect an uncomplicated behaviour of the flow of ξ_H on the singular set, however the author knows of no example that exhibits positive topological entropy (in [18] there are examples of geodesic flows on surfaces with genus ≥ 2 that possess a first integral independent of the energy integral on an open set X' but X' is not even dense).

For a real analytic completely integrable Hamiltonian it is reasonable to expect the vanishing of the topological entropy on the singular set. This is the case if the symplectic manifold has dimension 4 (this a consequence of results of Katok and Moser cf. [23]), however as far as we know the general case remains open.

Let us now describe our results. Let X be a symplectic $2n$-dimensional manifold, let H be a completely integrable smooth Hamiltonian and let $F = (f_1, ..., f_n)$. For each point $x \in X$ consider $\mathcal{O}(x)$, the orbit of x under the flows of $\xi_{f_1}, ..., \xi_{f_n}$ and suppose that the rank of $\mathrm{d}F_x$ is k. Then without loss of generality we can assume that $\xi_{f_1}, ..., \xi_{f_k}$ are linearly independent at x (and thus they will be linearly independent for every point in $\mathcal{O}(x)$). Associated with $\mathcal{O}(x)$ we can define a Lie algebra $\mathcal{F}$ as follows:

$$\mathcal{F} = \{g \in C^\infty(\mathcal{O}(x)) : \{f_i, g\} = 0, \quad i = 1, ..., n\},$$

where $C^\infty(\mathcal{O}(x))$ denotes the set of all smooth functions that are defined on some neighborhood of $\mathcal{O}(x)$. One easily checks that any two functions in $\mathcal{F}$ are in involution [15, Prop. 2.1] (in the latter reference the setting is real analytic but analyticity is not needed for the previously mentioned property).

Since $\xi_{f_1}, ..., \xi_{f_k}$ are linearly independent at x we can introduce [1] symplectic coordinates (u, v, z) with $(u, v) \in \mathbb{R}^{2k}$ and $z \in \mathbb{R}^{2m}$ $(m + k = n)$ in a neighborhood U of $x = (0, 0, 0)$ such that in these coordintes we have:

$$f_i(u, v, z) = v_i \quad i = 1, ..., k,$$

$$\omega = \sum_{i=1}^{k} \mathrm{d}u_i \wedge \mathrm{d}v_i + \sum_{i=1}^{m} \mathrm{d}\zeta_i \wedge \mathrm{d}\eta_i, \quad z = (\zeta, \eta).$$

With respect to these coordinates a function g belongs to $\mathcal{F}$ if and only if g is independent of u. For the following definition see also [15, Def. 2.3].

Definition 2.1. The orbit $\mathcal{O}(x)$ is said to be *non-degenerate* if there exists $g \in \mathcal{F}$ such that $g_z(0, 0) = 0$ and $g_{zz}(0, 0)$ is an invertible matrix.

We will say that H is completely integrable with *non-degenerate* first integrals if every orbit $\mathcal{O}(x)$ is non-degenerate in the sense we defined above.

The definition is independent of the choice of coordinate system since it can be reformulated in a more intrinsic language as follows. Set $G := (f_1, ..., f_k)$ and consider locally the symplectic reduction: $\pi : G^{-1}(h) \cap U \to (G^{-1}(h) \cap U)/\mathbb{R}^k$, where $G(x) = h$. Then $\mathcal{O}(x)$ is a non-degenerate orbit if and only if for some

$g \in \mathcal{F}$, $\xi_{\tilde{g}}$ has a non-degenerate zero at $\pi(x)$, where $\tilde{g}$ denotes the function induced by g on the symplectic reduction.

We also observe that Definition 2.1 is void at points $x \in X$ for which F is a submersion, hence non-degeneracy is in fact a condition on the first integrals over the singular set. One can regard non-degeneracy as a fairly "generic assumption" on the first integrals. In fact most of the well known examples verify this condition. For instance it is easy to check that non-degeneracy holds in the case of the spherical pendulum, the Kowalevskaya top and the Poisson sphere. The geodesic flow on the n-dimensional ellipsoid and the Neumann problem also admit non-degenerate first integrals; this is basically proved in [17, Section 6].

Our first result is the following:

Theorem 2.2. *Let X be a symplectic manifold and let H be a smooth Hamiltonian. Assume H is completely integrable with non-degenerate first integrals. Let ξ_H be the Hamiltonian vector field corresponding to H and let ψ_t denote the flow of ξ_H. Let $N \subseteq X$ be any compact separable ψ_t-invariant subset. Then the topological entropy of the flow $\psi_t\,|_N$ on N is zero.*

Hence, from our discussion in the introduction follows:

Corollary 2.3. *Let M be a compact Riemannian manifold whose geodesic flow is completely integrable with non-degenerate first integrals. Then $\pi_1(M)$ has sub-exponential growth. If $\pi_1(M)$ is finite then the loop space homology of M, $\sum_{i=1}^{k} \dim H_i(\Omega M, K)$, grows sub-exponentially for any coefficient field K.*

We should mention that regarding $\pi_1(M)$ and $H_1(M, \mathbb{Q})$ of a compact manifold M with a completely integrable geodesic flow, some very satisfactory results have been obtained by Taimanov [24,25] using different methods from ours. He showed in [24] that for real analytic integrable geodesic flows (and more generally for smooth geodesic flows with a set of "geometrically simple" smooth first integrals), $\pi_1(M)$ is almost abelian and $\dim H_1(M, \mathbb{Q}) \leq \dim M$. Moreover, in [25] it is shown that the rational cohomology ring of M contains a subring which is isomorphic to the rational cohomology ring of a torus of dimension equal to $\dim H_1(M, \mathbb{Q})$. We refer to [10] for related results for other mechanical systems.

Following Félix, Halperin and Thomas [9], we will say that a simply connected compact manifold M is *$\mathbb{Z}$-elliptic* if $\sum_{i=1}^{k} \dim H_i(\Omega M, K)$ grows polynomially for any coefficient field K. Spaces which are $\mathbb{Z}$-elliptic possess remarkable topological properties as it is shown in [9]. In general it is not known whether sub-exponential growth implies polynomial growth for the loop space homology unless the field K has characteristic zero or the characteristic $p \neq 0$ verifies the condition $p > \dim M/r$ where r is the least positive integer such that M

has homology in degree r [14]. As a result we do not know if Corollary 2.3 as well as our previous results in [21,22] can be strengthened to prove that in fact M is $\mathbb{Z}$-elliptic. Nevertheless we will show that $\mathbb{Z}$-ellipticity is a property that must be verified by those manifolds whose geodesic flows admit on a "nice" set action–angle coordinates with singularities.

Let X be a symplectic manifold of dimension $2n$, and let $H : X \to \mathbb{R}$ be a smooth Hamiltonian.

Definition 2.4. A system of action–angle coordinates with singularities around a point $x \in X$ for the Hamiltonian H, consists of an open set $U \subseteq X$ around x and a diffeomorphism $\phi : U \to T^k \times D_1 \times D_2$ where T^k is a k-dimensional torus with coordinates $\theta = (\theta_1, ..., \theta_k)$, D_1 is a domain of $\mathbb{R}^k$ containing the origin with coordinates $I = (I_1, ..., I_k)$ and D_2 is a domain of $\mathbb{R}^{2m}$ containing the origin with coordinates $(\zeta, \eta) = (\zeta_1, ..., \zeta_m, \eta_1, ..., \eta_m)$, such that

(i) $\omega = \phi^*\omega_0$ where $\omega_0 = \sum_{j=1}^{k} d\theta_j \wedge dI_j + \sum_{j=1}^{m} d\zeta_j \wedge \eta_j$, $(k + m = n)$.

(ii) $H \circ \phi^{-1}$ is a function only of $I_1, ..., I_k$ and $\tau_1, ..., \tau_m$ with $\tau_j = (\zeta_j^2 + \eta_j^2)/2$.

The classical action–angle coordinates correspond to $m = 0$. Eliasson, Ito and Dufour and Molino proved the existence of action–angle coordinates with singularities for completely integrable systems of "elliptic type" (cf. [8, p. 9] [15, Theorem 2.8] [7, p. 154] for details).

In Section 4 we will show:

Theorem 2.5. *Let M be a compact simply connected Riemannian manifold. Suppose there exists $p \in M$ such that for every vector in the unit sphere at p we can introduce action–angle coordinates with singularities for the geodesic flow. Then M is $\mathbb{Z}$-elliptic.*

3. Vanishing of the topological entropy

Let X be a symplectic manifold and let H be a smooth Hamiltonian. Assume that H is completely integrable with non-degenerate first integrals $f_1, ..., f_n$. Set $F = (f_1, ..., f_n)$ and define

$$\Sigma_k = \{x \in X : \operatorname{rank} dF_x = k\}.$$

Lemma 3.1. *Σ_k is a symplectic submanifold of dimension $2k$.*

Proof. Fix $x \in \Sigma_k$ and consider as in Section 2 a symplectic coordinate system (u, v, z) defined on a neighborhood U of x. It suffices to show that in these coordinates $U \cap \Sigma_k = \{(u, v, z) : z = 0\}$. Observe first that $(u, v, z) \in \Sigma_k$

if and only if every $g \in \mathcal{F}$ verifies $g_z(v,z) = 0$. From [15, Prop. 4.1] (no analyticity is needed for this claim) it follows that if the orbit is non-degenerate, any $g \in \mathcal{F}$ verifies $g_z(v,0) = 0$ and thus $\{(u,v,z) : z = 0\} \subseteq U \cap \Sigma_k$.

Now let $\widehat{g} \in \mathcal{F}$ be a function so that $\widehat{g}_z(0,0) = 0$ and $\widehat{g}_{zz}(0,0)$ is invertible. Hence it is clear that $U \cap \Sigma_k \subseteq \{(u,v,z) : \widehat{g}_z(v,z) = 0\}$, but since $\widehat{g}_{zz}(0,0)$ is invertible, the set $\{(u,v,z) : \widehat{g}_z(v,z) = 0\}$ is a smooth submanifold of dimension $2k$ in a neighborhood of the origin. Thus shrinking the set U if necessary, $U \cap \Sigma_k = \{(u,v,z) : z = 0\}$ as desired. □

Observation 3.2. For each point $x \in X$ consider the vector space $R(x)$ obtained by taking the quotient of Ker $\mathrm{d}F_x$ by the subspace generated by $\xi_{f_1}(x), ..., \xi_{f_n}(x)$. Now observe that the 2-jets of the first integrals whose differential vanishes at x, give rise to an abelian subalgebra $\mathcal{H}(x)$ of the space of quadratic forms on $R(x)$. In the local coordinates we introduced in Section 2, $\mathcal{H}(x)$ is generated by $(f_{k+1})_{zz}(0,0), ..., (f_n)_{zz}(0,0)$. One easily checks that if $\mathcal{H}(x)$ is a Cartan (i.e. maximal) subalgebra of the space of quadratic forms on $R(x)$ then the orbit $\mathcal{O}(x)$ is non-degenerate. Conditions of this sort were studied by Eliasson [8] and were also considered by Desolneaux-Moulis [5].

Before we start with the proof of Theorem 2.2 let us recall a few known facts. Let X be a symplectic $2n$-dimensional manifold, let H be a completely integrable smooth Hamiltonian and let $F = (f_1, ..., f_n)$. Let I_c^0 denote some connected component of $F^{-1}(c)$. Then if c is a regular value of F and ψ_t denotes the flow of ξ_H, then $\psi_t|_{I_c^0}$ is differentiably conjugate to a translation-type flow on $\mathbb{R}^k \times T^{n-k}$ [1, Theorem 5.2.24]. A flow $\varphi_t : \mathbb{R}^k \times T^{n-k} \to \mathbb{R}^k \times T^{n-k}$ is called a *translation-type flow* if for some vector $\nu = (\nu_1, ..., \nu_n) \in \mathbb{R}^n$ we can write

$$\begin{aligned}&\varphi_t(x_1, ..., x_k, \theta_{k+1}, ..., \theta_n)\\ &\quad = (x_1 + t\nu_1, ..., x_k + t\nu_k, \theta_{k+1} + t\nu_{k+1} (\text{mod } 1), ..., \theta_n + t\nu_n (\text{mod } 1)).\end{aligned} \tag{1}$$

Clearly for $k = 0$ the flow is periodic or quasi-periodic.

Recall that an orbit γ of a flow ψ_t is called *recurrent* if $\gamma \subseteq \omega(\gamma)$, where $\omega(\gamma)$ denotes the ω-limit set of γ. Clearly it follows from the definition that a translation-type flow has no recurrent orbits if $k \neq 0$, i.e., if it acts on a non-compact space (provided of course, one does not consider the trivial case $\nu_1 = ... = \nu_k = 0$ in which the first k coordinates can be ignored).

Proof of Theorem 2.2. Using the variational principle of the topological entropy it suffices to show that if μ is a Borel ψ_t-invariant ergodic probability measure on N then the metric entropy of the system $(N, \psi_t|_N, \mu)$ is zero. Consider as before the strata Σ_r as

$$\Sigma_r = \{x \in X : \text{ rank d}F_x = r\}.$$

Since Σ_r is ψ_t-invariant for all $0 \leq r \leq n$ and μ is ergodic, there exists r_0 and a connected component of Σ_{r_0} that we call $\Sigma^0_{r_0}$, for which $\mu(\Sigma^0_{r_0} \cap N) = 1$. We can assume without loss of generality that $df_1, ..., df_{r_0}$ are linearly independent *everywhere* on $\Sigma^0_{r_0}$. Let $G := (f_1, ..., f_{r_0})|_{\Sigma^0_{r_0}} : \Sigma^0_{r_0} \to \mathbb{R}^{r_0}$. Once again by ergodicity there exists $c \in \mathbb{R}^{r_0}$ so that for some connected component, I^0_c, of $G^{-1}(c)$ we have that $\mu(I^0_c \cap N) = 1$. By the Poincaré Recurrence Theorem [19, Theorem 2.3] the set $\mathcal{R}$ of recurrent orbits of ψ_t on $I^0_c \cap N$ has full measure. Hence it suffices to show that the metric entropy of the system $(\mathcal{R}, \psi_t|_{\mathcal{R}}, \mu)$ is zero.

Now by Lemma 3.1 if H is completely integrable with non-degenerate first integrals, then the set $\Sigma^0_{r_0}$ is a symplectic submanifold of X of dimension $2r_0$. Thus the map $G : \Sigma^0_{r_0} \to \mathbb{R}^{r_0}$ is a submersion and then it follows from the discussion after Observation 3.2 that $\psi_t|_{I^0_c}$ is differentiably conjugate to a translation-type flow on $\mathbb{R}^k \times T^{r_0-k}$. But if $k \neq 0$ it follows that ψ_t has no recurrence on I^0_c and thus $\mathcal{R}$ is empty which is absurd. Hence $k = 0$ and the flow ψ_t on I^0_c is periodic or quasi-periodic. In either case it is well known that the metric entropy of such flow with respect to any invariant probability measure is zero and thus the metric entropy of the system $(\mathcal{R}, \psi_t|_{\mathcal{R}}, \mu)$ is zero, as desired.

□

4. $\mathbb{Z}$-ellipticity

We briefly describe the idea of the proof of Theorem 2.5. Via Morse theory we can bound the growth of the loop space homology by the growth of the volume of the iterates under the geodesic flow of the unit sphere at p. The existence of action–angle coordinates with singularities will imply that the latter growth is polynomial.

We start with a preliminary discussion. Let M be a compact Riemannian manifold with geodesic flow ψ_t. Let S_p denote the unit sphere at p with canonical measure ν. For each $q \in M$ define $n_{p,\lambda}(q)$ as the number of geodesics connecting p and q with length $< \lambda$. Consider the function

$$I_p(\lambda) = \int_M n_{p,\lambda}.$$

In [3] it was proved that this integral is well defined and

$$I_p(\lambda) = \int_0^\lambda \mathrm{d}t \int_{S_p} |\det A_v(t)| \mathrm{d}\nu(v), \tag{2}$$

where $A_v(t)\colon (\psi_t v)^\perp \to (\psi_t v)^\perp$ is the unique family of linear maps along the geodesic defined by v verifying the Jacobi equation with initial conditions $A_v(0) = 0$ and $A_v'(0) = \mathrm{I}d$.

For each $v \in S_p$ take the geodesic $exp_v tv$ and an orthonormal basis $\{v, e_2, ..., e_n\}$ in T_pM. Consider the Jacobi fields $J_i(t)$ such that $J_i(0) = 0$, $J_i'(0) = e_i$. Then $|\det A_v(t)| = \sqrt{|\det\langle J_i(t), J_j(t)\rangle|}$. Since

$$\sqrt{|\det\langle J_i(t), J_j(t)\rangle|} \leq \| J_2(t) \| ... \| J_n(t) \|$$

and $\| J_i(t) \| \leq \| \mathrm{d}\psi_t \|$ we obtain from equation (2)

$$I_p(\lambda) \leq \int_0^\lambda \mathrm{d}t \int_{S_p} \| \mathrm{d}\psi_t \|^{n-1}\,\mathrm{d}\nu. \tag{3}$$

On the other hand, it follows from Morse theory and results of Gromov in [11] (cf. also [21, p. 115]), that if $\pi_1(M)$ is finite there exists a constant $c > 0$ depending only on the geometry of M so that for $\lambda \geq 1$:

$$\mathrm{Vol}(M) \sum_{i=1}^{c(\lambda-1)} \dim H_i(\Omega M, K) \leq I_p(\lambda).$$

Hence if we combine the last equation with equation (3) we obtain

$$\sum_{i=1}^{c(\lambda-1)} \dim H_i(\Omega M, K) \leq \frac{1}{\mathrm{Vol}(M)} \int_0^\lambda \mathrm{d}t \int_{S_p} \| \mathrm{d}\psi_t \|^{n-1}\,\mathrm{d}\nu. \tag{4}$$

Proof of Theorem 2.5. Equation (4) implies that to prove Theorem 2.5 it suffices to show that $\| d\psi_t \|$ is uniformly bounded for each $v \in S_p$ by a polynomial in t.

Fix $v \in S_p$ and let U and ϕ be as in Definition 2.4. Our setting now is $X = TM$ with the symplectic structure obtained by pulling back the canonical symplectic structure of T^*M by using the Riemannian metric and $H = \frac{1}{2}\|v\|^2$.

Conditions (i) and (ii) in Definition 2.4 imply that the flow $\phi \circ \psi_t \circ \phi^{-1}$ of $\xi_{H\circ\phi^{-1}}$ is determined by the equations $(k + m = n)$:

$$\dot{\theta}_j = \frac{\partial(H\circ\phi^{-1})}{\partial I_j}, \quad \dot{I}_j = 0 \quad (j = 1, ..., k),$$

$$\dot{\zeta}_j = \frac{\partial(H\circ\phi^{-1})}{\partial\tau_j}\eta_j, \quad \dot{\eta}_j = -\frac{\partial(H\circ\phi^{-1})}{\partial\tau_j}\zeta_j \quad (j = 1, ... m).$$

These equations are easily solved and the following formula for the flow of $\xi_{H\circ\phi^{-1}}$ is obtained:

$$\begin{aligned} &\phi \circ \psi_t \circ \phi^{-1}(\zeta, \eta, I, \theta) \\ &= (\zeta\cos(\omega_1 t) + \eta\sin(\omega_1 t), \eta\cos(\omega_1 t) - \zeta\sin(\omega_1 t), I, \omega_2 t (\mathrm{mod}\ 1) + \theta), \end{aligned} \tag{5}$$

where

$$\omega_1(I,\tau) = \frac{\partial(H\circ\phi^{-1})}{\partial\tau}(I,\tau),$$
$$\omega_2(I,\tau) = \frac{\partial(H\circ\phi^{-1})}{\partial I}(I,\tau). \tag{6}$$

Thus the differential $d\phi\circ d\psi_t\circ d\phi^{-1}$ can be represented by a matrix whose entries are bounded by linear polynomials in t. Hence there exist constants a and b (that depend on U) so that for $v \in U$

$$\|d\psi_t\| \leq at + b. \tag{7}$$

Since S_p is compact and we can introduce action–angle coordinates with singularities for every point in S_p, we can cover S_p with a finite number of sets for which the bound (7) holds and thus $\| d\psi_t \|$ can be bounded uniformly in S_p by a linear polynomial in t as desired.

□

5. Final remarks

Corollary 2.3 also holds for a more general class of Hamiltonian systems on cotangent bundles. It is sufficient to assume that H is optical (or convex) i.e. H has positive definite Hessian restricted to the fibres of T^*M. For such Hamiltonians the vanishing of the topological entropy of the flow restricted to an appropriate energy level imposes the same topological restrictions on the loop space homology as for the geodesic flow case.

It would be very interesting to investigate under which conditions a completely integrable Hamiltonian can be approximated by another with non-degenerate first integrals. A statement of this sort, combined with the fact that optical Hamiltonians are rough, will imply that Corollary 2.3 also holds for a much larger class of systems. This idea was in fact what motivated the author to consider non-degenerate systems in the sense of Definition 2.1.

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